Marquette University
2014

COMPETITIVE SCHOLARSHIP EXAMINATION
IN
MATHEMATICS

Do not open this booklet until you are directed to do so.

1. Fill out completely the following information about yourself.

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<th>Last name</th>
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Your high school: Name ______________________ City ______________________

High School Counselor or Advisor: __________________________________________

2. This examination consists of two parts. The time allowed for each will be approximately 60 minutes. Should you finish Part I early, you may proceed to Part II.

3. Part I consists of 20 objective-type questions. Each question has five possible answers marked: A., B., C., D., E. Only one answer is correct. You are to circle the letter corresponding to the correct response for as many problems as you can.

Example: If \( x = 5 \) and \( y = -2 \), then \( x + 4y \) is

A. \(-3\)  B. \(-2\)  C. \(-1\)  D. 0  E. \(+1\).

4. Part II consists of 3 subjective-type questions. Show a summary of your work in this booklet for each question you attempt, whether or not you obtain a complete solution. Scratch paper is provided but be sure to show the essential steps of your work concisely in the space provided for each question. Only the work appearing in this booklet will be scored. You will be scored on your method of attack, ingenuity, insight, inventiveness, and logical developments as well as your solutions.

5. Pencils and scratch paper will be provided. No tables, rulers, compasses, protractors, slide rules, calculators, or other aids are permitted.

6. a. The scoring of questions in Part I has been devised to discourage random guessing and will be computed as follows:

   (three times number correct) - (number wrong).

b. The scoring for the three questions in Part II will be 13, 13, and 14 for a total of 40 points. Partial credit will be given so it will be to your advantage to do as much as you are able to do on each question.

7. For the scoring committee. Do not write in the box below.
PART I

1. If $2, a, \frac{1}{2}$ are consecutive terms in a geometric progression, then

(A) $a$ is an positive fraction.

(B) $a = \frac{2}{\sqrt{2}}$

(C) $a = 1$

(D) $a$ is not uniquely defined

(E) $a = -\sqrt{2}$

2. The line $\ell : y = ax + b$ is parallel to the line $3y + 2x + 1 = 0$ and contains the point $(1, 1)$. Then $\ell$ intersects the $x$-axis in the point

(A) $\left(0, \frac{5}{2}\right)$

(B) $\left(0, \frac{2}{3}\right)$

(C) $\left(\frac{3}{2}, 0\right)$

(D) $\left(-\frac{5}{2}, 0\right)$

(E) $\left(\frac{5}{2}, 0\right)$

3. A circle of radius 2 and center $C$ intersects the line $\ell$ in the points $P$ and $Q$. If the length of $PQ$ is twice the distance of $C$ from the line $\ell$ then length of $PQ$ is

(A) $2\sqrt{2}$

(B) $\sqrt{2}$

(C) $\frac{1}{2}$

(D) 1

(E) 2

4. If $1 + \frac{1}{1 + \frac{1}{x}} = \frac{1}{2}$, then $x$ equals

(A) $\frac{1}{3}$

(B) $-\frac{1}{3}$

(C) 2

(D) $\frac{1}{2}$

(E) 3
5. If $|−a + b + c| = |a − b + c| = |a + b − c|$, then
   (A) $a + b + c = 0$
   (B) $abc ≠ 0$
   (C) $|a + b + c| = |a| + |b| + |c|$
   (D) $a + b + c ≠ 0$
   (E) $abc = 0$

6. Randomly choose two digits from 0, 1, 2, ..., 9. The probability that both digits add up to 5 is
   (A) $\frac{1}{10}$
   (B) $\frac{1}{5}$
   (C) $\frac{1}{15}$
   (D) $\frac{2}{15}$
   (E) $\frac{1}{45}$

7. If $f(x) = \frac{x}{1 + x^2}$ and $x ≠ 0$ then $f\left(-\frac{1}{2}\right)$ is equal to
   (A) $f(x)$
   (B) $−f(x)$
   (C) $\frac{1}{f(x)}$
   (D) $−\frac{1}{f(x)}$
   (E) $−\frac{f(1)}{f(x)}$

8. Let $a$ and $b$ be positive real numbers and $a ≠ 1$, $b ≠ 1$, such that $\log_a b = \frac{1}{2}$. Then
   (A) $a > 1$ and $b < 1$
   (B) $a > 1$ and $b > 1$
   (C) $\log_a b − \log_b a > 0$
   (D) $\log_a b + \log_b a = \frac{5}{2}$
   (E) $\log_b a < 0$
9. If \( x + y = 3 \) and \( x^3 + y^3 = 63 \), then \( xy \) equals

(A) 21
(B) -21
(C) 27
(D) -4
(E) 2

10. If \( a + b = \frac{1}{a - b} \), then

(A) \( a > b \)
(B) \( b > a \)
(C) \( a + b \neq 0 \)
(D) \( a = b \)
(E) \( |a + b| = a + b \)

11. The circle \( \left( x - \frac{3}{2} \right)^2 + y^2 = 4 \) and the ellipse \( 4x^2 + 9y^2 = 1 \) have \( n \) points in common. Then \( n \) equals

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4

12. There exists a set of 20 consecutive integers whose sum is

(A) 0
(B) odd
(C) not divisible by 5
(D) divisible by 19
(E) divisible by 20
13. \( M, N, P \) are the midpoints of the sides of the triangle \( ABC \), and \( E, F, G \) the midpoints of the sides of the triangle \( MNP \). If the perimeter of the triangle \( EFG \) is \( n \) times the perimeter of the triangle \( ABC \), then \( n \) equals

(A) 4
(B) \( \frac{1}{4} \)
(C) 2
(D) \( \frac{1}{2} \)
(E) \( \frac{1}{8} \)

14. If \( X + 1 \) divides \( X^5 + 2X^4 + 3X^3 + aX^2 + 5X + 6 \), then \( a \) equals

(A) 5
(B) 4
(C) 3
(D) 2
(E) 1

15. If \( \tan \frac{a}{2} = 1 \), then \( \sin a \) equals

(A) 0
(B) 1
(C) \(-1\)
(D) \( \frac{1}{2} \)
(E) \(-\frac{1}{2}\)

16. \( a \) and \( b \) are positive integers such that \( ab = 96 \) and the greatest common divisor of \( a \) and \( b \) is 2. Then \( a + b \) equals

(A) 20 or 52
(B) 22 or 50
(C) 16
(D) 48
(E) 24
17. If a circle of radius 1 is tangent to all four sides of a rhombus, and the area of the rhombus is twice the area of the circle, then each side of the rhombus has length

(A) 1
(B) 2
(C) \( \frac{1}{2} \)
(D) \( \frac{3}{2} \)
(E) \( \pi \)

18. If \( X_1 \) and \( X_2 \) are the two solutions of \( X^2 + 15X + 1 = 0 \), then the solutions of \( X^2 - 15X + 1 = 0 \) are

(A) \(-X_1 \) and \( \frac{1}{X_1} \)
(B) \( X_2 \) and \( \frac{1}{X_2} \)
(C) \( X_1 \) and \( \frac{1}{X_1} \)
(D) \(-X_2 \) and \( \frac{1}{X_2} \)
(E) \( X_1 \) and \(-X_2 \)

19. The radius of the circumscribed circle of an equilateral triangle is \( n \) times the radius of its inscribed circle. Then \( n \) equals

(A) \( \sqrt{2} \)
(B) \( \sqrt{3} \)
(C) 2
(D) 3
(E) \( \frac{3}{2} \)

20. If \( n \) is the smallest odd integer such that \( n - 3 \) divides \( 5n - 3 \), then

(A) \( n = 9 \)
(B) \( n = 5 \)
(C) \( n = 1 \)
(D) \( n > 3 \)
(E) \( n < -5 \)
PART II

1. A triangle $ABC$ has a right angle at $A$, and $P$ is a point on the side $AB$ such that $PB + BC = PA + AC$. Also $\frac{PB}{PA} = \frac{3}{7}$ and $AC = 21$. 

   (a) Find the area and the perimeter of the triangle $ABC$.

   (b) There exists a point $I$ inside the triangle $ABC$ whose distance to each of the sides is the same. Find that distance.
2. A decimal fraction is a fraction whose denominator is a power of 10.  

(a) Find a decimal fraction which equals \( \frac{13}{64} \).

(b) Find the smallest positive integer \( m \) such that \( \frac{m}{5m - 1} < \frac{13}{64} \) and \( \frac{m}{5m - 1} \) equals a decimal fraction.
3. A cube whose edges have length 1 is given; \(\text{14 POINTS}\)

\((a)\) Find the radius of the sphere that passes through all 8 vertices of the cube.

\((b)\) Find the radius of the sphere that passes through the four vertices of one face of the cube, and is tangent to the opposite face.